Granular relaxation under tapping and the traffic problem

D.C. Hong,[1] S. Yue,[1] J.K. Rudra,[2] M.Y. Choi,[3] and Y.W. Kim[1]
[1]*Department of Physics, Lehigh University, Bethlehem, Pennsylvania 18015*
[2]*Department of Physics, Xavier University of Louisiana, Post Box 116c, 7325 Palmetto Street, New Orleans, Louisiana 70125*
[3]*Department of Physics, Seoul National University, Seoul 151-742, Korea*

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We study the relaxation of a one-dimensional granular pile of height L in a confined geometry under repeated tapping within the context of the diffusing void model. The reduction of height as a function of the number of taps is proportional to the accumulated void density at the top layer. The relaxation process is characterized by the two dynamic exponents z and z' which describe the time dependence of the height reduction $\Delta h(t) \approx t^z$ and the total relaxation time $T(L) \approx L^{z'}$. While the governing equation is nonlinear, we find numerically that $z = z' = 1$, which is robust against perturbations and independent of the initial void distributions. We then show that the existence of a steady state traveling wave solution is responsible for such a linear behavior. Next, we examine the case where each void is able to maintain its overall topology as a round object that can subject itself to compression. In this regime, the governing equations for voids reduce to traffic equations and numerical solutions reveal that a cluster of voids arrives at the top periodically, which is manifested by the appearance of periodic solutions in the density at the top. In this case, the relaxation proceeds via a stick-slip process and the reduction of the height is sudden and discontinuous.

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I. INTRODUCTION

One of the difficulties in studying the dynamic response of granular assembly is our inability to derive any sensible continuum equation starting from the first principle [1]. This is in contrast with what we have encountered in fluid mechanics, where any fluid instability can be, in principle, studied by careful examination of the Navier-Stokes equation. For this reason, much of the current activity in granular dynamics has focused on large scale computer simulations such as molecular dynamics studies, where the interparticle interactions and friction laws were put in to mimic the realistic situations [2–6], or cellular automata [7, 8], where dynamics must be put in by hand based on some physical arguments. While there seem to exist some fundamental differences between ordinary fluid and the granular systems, for example, such as the absence of precise relation between the stress and the rate of strain in granular materials, at least at the level of molecular dynamics or cellular automata currently employed in the literature, the primary difference appears to lie in *density.* It is not very clear whether the form of interparticle potential is really crucial in producing several distinctive nonlinear responses of granular assembly, which are absent in fluids, via molecular dynamics. Even with elaborate friction laws [8], it is equally unclear whether such unusual responses would persist in the low density limit. If so, then the simple hard sphere gas with perhaps suitable friction laws should be able to produce most of the current molecular dynamics results such as Brazilian nut segregation [9–13], convection [14–16], propagating density waves [17–20], and possibly, segregation of two different types of grains under rotation [21]. Dynamics of hard sphere gas is controlled by the momentum transfer through a direct contact. Hence, in the spirit of hard sphere gas, it is not unphysical to assert that tracing the continuous yet stochastic rearrangement of grains to external stimuli is one of the key elements in understanding the complex dynamic response of the granular assembly. With this in mind, the diffusing void model [22] was proposed to trace the stochastic movement of voids rather than the grains under gravity. The thermodynamic model of Mehta and Edwards [23], which is based on the random packing of grains, might be viewed from this aspect, too, even though it is an equilibrium model.

The diffusing void model has been shown to produce most of the unique features of the *granular flow* patterns in a confined geometry in the slow limit. The purpose of this paper is to extend the diffusing void model further to study the *dynamic response* of the granular assembly. Our primary focus in this paper is to understand the recent experiment performed by Jaeger and his collaborators [24], where a granular pile in a tube is repeatedly tapped and the relaxation of the height was investigated. We will first explain how one can study this problem within the context of the diffusing void model, then derive the dynamic equation of motion for the void, and present numerical solutions along with a particular solution relevant to the present case. We will further examine under what conditions the equation should be modified and show how the appropriate change of the diffusing void model allows us to map this problem onto the traffic problem, which predicts quite interesting relaxation behavior.

This paper is organized as follows: Section II presents the diffusing void model extended properly to study the dynamic response. We begin with the continuity equa-

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tion for the void density, and consider the fact that the motion of a void is affected by the presence of voids above it. This allows us to write the upward velocity of a void as a function of the void density and its gradient, and leads to the dynamic equation for the voids. The obtained equation is solved numerically in Sec. III for various initial void distributions. We present one particular solution to the dynamic equation, which has the form of the traveling wave, in an attempt to explain the numerical results. Section IV is devoted to the modification of the dynamic equation by explicitly writing down the time dependent equation for the void velocity, which manifests its close relation to the traffic problem. Finally, a summary is given in Sec. V.

II. DIFFUSING VOID MODEL

Our investigation is based on the model recently proposed in an attempt to study the dynamics of grains in a confined geometry [25, 26]. This is a continuum version of the discrete random walk model of granular flow [22], termed "the diffusing void model." The model is based on the assumption recognized previously by Litwinyszyn [27] and Mullins [28] and others [29] that the flow of granular particles in a confined geometry is caused by the upward motion of voids resulting from the escape of granular particles through an orifice. But there are several crucial differences in our approach: The model can describe the evolution of the free surface, stream lines with and without obstacles, shock front below the obstacle, and stagnant solids, which the previous authors were unable to handle. This model was shown to correctly reproduce most of the unique features of granular flows in a confined geometry, in particular, the flow patterns and stream lines in any geometry with and without obstacles. In addition, we have recognized that the cascading process at the surface and the motion of grains at or near the boundary and obstacles are intrinsically nonlinear and hence the equation of motion must be nonlinear, which is, in our opinion, a sharp departure from the previous authors [27–29] who mainly used the linear biased diffusion equation.

We now apply and extend this model further to study the relaxation of granular particles to tapping. When the grains confined in a tube are subject to tapping, the height relaxes. Hence, the scientific question we address in this paper is simple and straight forward: how does the height of the granular assembly decrease as a function of the number of taps or, in case of continuous tapping, of time? Note that in any granular assembly the grains are never perfectly ordered and an enormous amount of minute voids are always present. Hence, if a vertical column of granular materials is repeatedly tapped, the grains become more ordered and the voids will move upward until they meet the sea of voids (empty space), which results in the reduction of the height. This process is schematically drawn in Fig. 1.

Hence, in this picture, the reduction of the height, $\Delta h(t) = h_0 - h(t)$ with $h_0 = L$ the initial height and $h(t)$ the height at time t, will be proportional to the accumulated void density at the top layer:

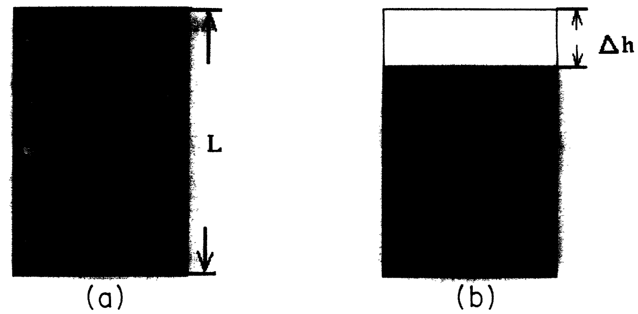

FIG. 1. Schematic picture of the relaxation of a granular pile under repeated tapping. (a) Grains are confined in a quasi-one-dimensional tube whose initial height is L. Since the grains (black) are randomly packed, there exist numerous minute voids inside (white circles). The top layer of the pile meets the sea of voids. (b) Under repeated tapping, grains relax and the voids move up, which are then accumulated at the top. The reduction of the height Δh is proportional to the total accumulated void density at the top layer [See Eq. (1)]. Note that the massive black also contains regular as well as irregular voids even in the totally relaxed configuration, which must be subtracted in the stricted sense.

$$\Delta h(t) \approx \rho_{\text{acc}} = \int_{h_0}^{h(t)} dz \rho(z,t) \sim t^z, \tag{1}$$

where we have defined the dynamic exponent z to characterize the relaxation process. A variable of particular relevance in experiments might be the saturation time T, beyond which the height saturates and the reduction of height no longer results in by tapping. The saturation time is the total relaxation time to the ground state. We might define the second dynamic exponent, z', to characterize this process,

$$T(L) \sim L^{z'}, \tag{2}$$

Note that since the dynamics is initiated by tapping, the time t is expected to be linearly proportional to the number of taps [30]. We point out here that even in the totally relaxed configuration, there still exist regular and irregular voids, which must be subtracted in the stricted sense. The voids defined here are coarse grained ones in such a way that in the totally relaxed configurations, there exist no voids at all. One might equally define the totally relaxed state as the one with some constant void fraction and stop simulations whenever the void fraction reaches this point.

We now derive the dynamic equation of motion for the voids. Note that the current density $\mathbf{J}$ of the voids is related to the time evolution of the void density $\rho(\mathbf{r},t)$ at any point via the continuity equation:

$$\frac{\partial}{\partial t}\rho(\mathbf{r},t) + \nabla \cdot \mathbf{J} = 0. \tag{3}$$

In the absence of gravity the grains in the assembly do not interact with each other and the thermal energy is too small to trigger the motion of grains. So we may rule out any simple density-gradient driven isotropic diffusion of the voids. However, unlike a liquid, under the influence of the gravitational field, the grains exert frictional forces on one another. Motion of the voids is, therefore, only due to the gravitational and the resulting frictional forces, which cause them to move upward (in the z direction) with velocity V_z and also horizontally (in the xy plane) with velocity V_{xy}. The granular flow is thus caused by "the biased diffusion" of the voids due to gravity.

If the granular assembly is confined in a box where the height is substantially larger than the width, the problem then becomes effectively a one-dimensional one and only the motion along the z direction is of relevance. For this one-dimensional problem, the current density J_z of the voids along the vertical direction can be written as a product of the velocity and the density

$$J_z = V_z \rho(z). \tag{4}$$

The upward velocity V_z is simply caused by the vertical push of the gravity. Note, however, that the motion of the voids has relevance only when they are inside the granular assembly. They stop moving whenever they face a free surface or the empty space (sea of voids). Thus a void ceases to move upward when there is a complete sea of voids (i.e., $\rho = 1$) immediately above it. Hence, $V_z(z)$ is expected to be proportional to $[1 - \rho(z+\Delta z)]$ with Δz the characteristic size of the void.

We therefore write the upward velocity V_z in the form

$$V_z = V_0 \{[1 - \rho(z+\Delta z]\} = V_0 \{[1 - \rho(z)]\} - D \frac{d\rho}{dz}, \tag{5}$$

where V_0 is assumed to be a constant and $D \equiv V_0 \Delta z$ has the dimension of the diffusion constant. One may treat Δz as a dynamical variable and write down a separate equation, but in the mean field limit(or in the first order approximation) which we are concerned here, it may be considered as a constant. Combining Eqs. (3), (4), and (5), we arrive at the dynamic equation for the voids:

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial z}[V_0 \rho (1 - \rho)] + D \frac{\partial}{\partial z}\left(\rho \frac{\partial \rho}{\partial z}\right), \tag{6}$$

the solution of which will provide information about the relaxation of grains. Note that the term $\rho(1 - \rho)$ has maximum at $\rho = 1/2$, which might be considered as the barrier that the void has to overcome. However, more rigorous analysis must be taken to acommodate the threshold condition, which is essential for failure dynamics such as granular dynamics, earthquakes, peeling, and dynamic fracture [31]. Since Eq. (6) is intrinsically nonlinear, we first pursue numerical solutions and then present one particular solution to the similar but a slightly different one, which is the subject of Sec. III.

III. NUMERICAL SOLUTIONS

In order to solve numerically Eq. (6), we first have to provide the initial conditions. We assume that the height of the column is L and that initially voids are distributed according to the various distribution functions. We then supply the two fixed boundary conditions, $\rho(L,t) = 1$ and $\rho(0,t) = 0$, which is simply the statement that the top of the pile meets the sea of voids and the bottom is supported by the grains. As we tap continuously, voids inside the pile diffuse out and pile up at the top of the column. Hence, the region with $\rho(z,t) = 1$ will increase. We now discretize the column into L positions with $\Delta z = 1$, for which case $D = V_0$. The discretized version of the time evolution equation (6) for the void density at position j takes the form

$$\frac{\partial \rho(j)}{\partial t} = -V_0\, \rho(j)\, [1 - \rho(j{+}1)] + V_0\, \rho(j{-}1)\, [1 - \rho(j)], \tag{7}$$

where we have symmetrized the second term in Eq. (6):

$$\frac{\partial}{\partial z}\left(\rho \frac{\partial \rho}{\partial z}\right) = \rho(j)\, [\rho(j{+}1) - \rho(j)] - \rho(j{-}1)\, [\rho(j) - \rho(j{-}1)],$$

with the boundary conditions $\rho(0) = 0$ and $\rho(L{+}1) = 1$. Equation (7) has a simple physical interpretation: The first term represents the loss due to the movement of a void from $z = j$ to $z = j + 1$, while the second term describes the gain resulting from the migration of a void from $z = j - 1$ to j. We now present numerical results for the relaxation with different initial void distributions.

(a) Random distribution. In this case, the initial voids distribution is random with a mean value 0.5. So, the total number of voids inside the tube is $L/2$, which is the maximum height reduction. In order to trace the relaxation process, we display the three representative snap shots of the time evolution of voids distribution in Fig. 2(a)–2(c). Note that in what follows, all the simulations are done with an initial void fraction given by $L/2$.

As the void diffuses out, it moves to the top and piles up. Now, once the void density becomes one, it remains one. The position of the top layer of the pile at time $t, h(t)$, can be easily identified as the interface where the density ρ drops sharply from one. This is the interface dividing the unrelaxed granular pile from the sea of voids. This interface moves down as the time progresses until all the voids inside the pile have diffused out. As pointed out before, the reduction of the height $\Delta h(t)$ at the time when the interface position is $h(t)$ is given by $\Delta h(t) = \int_{h_0}^{h(t)} dz$ since $\rho(z,t) = 1$ for $h(t) \leq z \leq h_0$. In Fig. 3(a) is plotted $\Delta h(t)$ as a function of time t for five different values of L: $L = 1000$, 3000, 5000, 7000, and 9000. In all cases, it is evident that $\Delta h(t)$ is *linear* in t. Since by a simple rescaling of parameters, the Eq. (6) can be recast into the form of (7), the linear relaxation should persist for different values of V_0 and D as long as the initial void distribution is random: $\Delta h(t) \approx \beta t^z$ with the dynamic exponent $z = 1$. In addition, the exponent $z = 1$ is independent of the total void fraction inside the pile. We have decreased the total void fraction from 0.5 to 0.2 and have found essentially no difference in the dynamic exponent z. However, as usual, the proportionality constant β is not universal and is linear in $\langle \rho_0 \rangle$, which is the average void density at time $t = 0$. We have also measured the total relaxation time, $T(L)$, as a

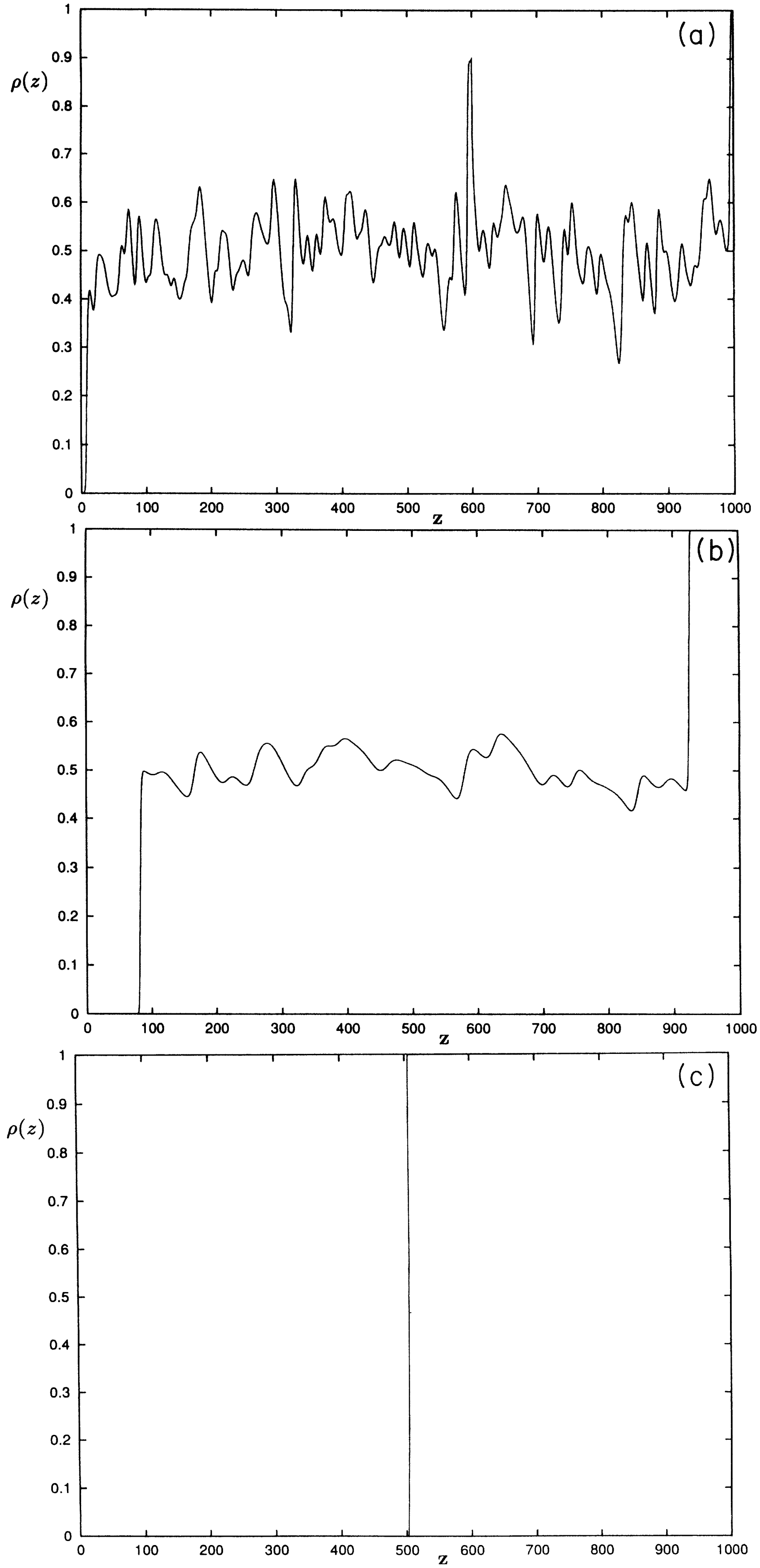

FIG. 2. Snap shots of the density distribution for three different times. (a) At t=0, the void distribution is random with mean value 0.5. The initial height of the pile L=1000. $z = L$ is the position of the interface between the top layer of the pile and the sea of voids. (b) At t=15. Voids move upward and accumulate near the top (the region where density is one) and the interface moves down. (c) At t=1500, all the voids have migrated upward resulting in the complete relaxation. Since initially there are $L/2$ voids, the total relaxation in the height is also $L/2$.

function of t and found that it is also linear [Fig. 3(b)]. Hence, the second dynamic exponent $z' = 1$.

We now consider the slight variations in the distribution function. Experimentally, it has been recognized that a loosely packed horizontal plane is often hard to remove when preparing the granular systems. In order to mimic this situation in our model, we have considered two cases. First, we select one layer, say $z = z_0$ and let the density at that layer remain fixed: $\rho(z_0, t) = \rho_0$. If $\rho_0 \gg 0$, this effectively models a case where a thin layer of void source is placed at $z = z_0$. If $\rho_0 \ll 1$, then we might say a barrier is placed at $z = z_0$. If we let the density at z_0 evolve according to the dynamic equation, then we would not expect any substantial effect of this layer to the relaxation process.(Note the distribution is random.) So, we have set the density at z_0 remain fixed during the course of dynamic evolution. Physically this means that we are placing a (weak) source or sink at $z = z_0$. We first consider the case with $\rho_0 = 0.05 \ll 1$. In this case, the relaxation process still remains linear, but the coefficients β crosses over to a different value as shown in Fig. 4(a), which is not unexpected. The movement of voids above the layer are not affected by the presence of the layer and hence the relaxation will proceed as if it were not there until all the voids above the layer have diffused out. After that, we expect the relaxation to be suppressed because the layer blocks the movement of voids. We have measured the crossover time, τ_c, as a function of the distance from the top layer to the barrier [Fig. 4(b)], which still exhibits the linear behavior. If we put a source term at z_0 with $\rho(z_0) > 0.5$, there exists no crossover time and the relaxation proceeds impressively linear all the way up

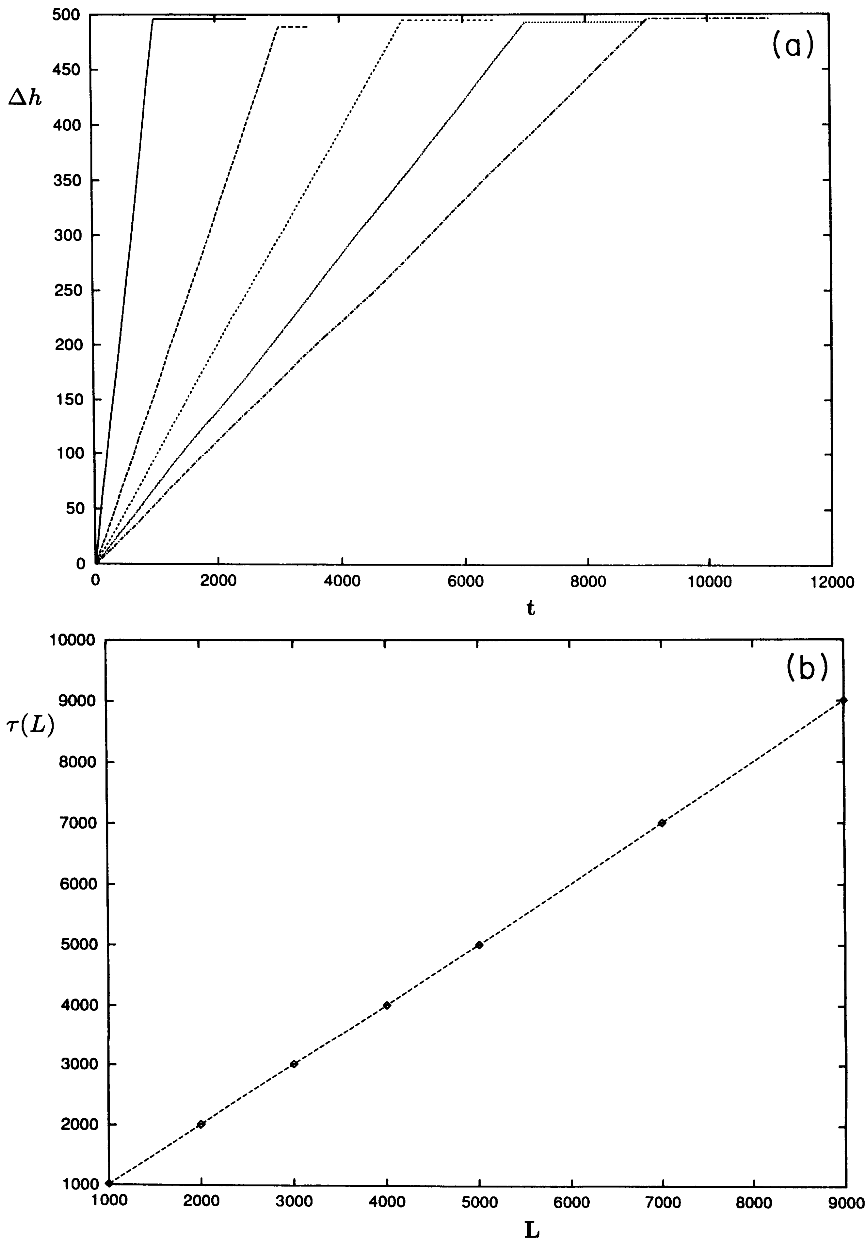

FIG. 3. (a) Reduction of the height $\Delta h(t)$ as a function of time t with the random initial distribution. Clockwise from the left, L=1000, 3000, 5000, 7000, and 9000. In all cases, the total relaxation occurs when the height becomes $L/2$ because initially voids fraction is 0.5. Note that the slope is one and thus the relaxation is linear in t : $\Delta h(t) \approx \beta t^z$ with z=1 and $\beta \approx \langle \rho_0 \rangle$. (b) Total relaxation time $T(L)$ as a function of L. The slope is 1.

to the total relaxation time.

In summary, we have shown that the dynamic exponents z and z' are robust for a given random distribution. They are not sensitive to the perturbations to the given initial distributions. The next obvious question is to examine whether such robustness persists for different distribution functions. To this end, we consider two different distributions: linear distribution, and the Boltzmann distribution, and examine the question of universality.

(b) Linear distribution. The initial voids distribution is given by, $\rho(z,0) = 1 - z/L$. More voids are at the bottom than the top. Note that $\int_0^L dz\rho(z,0) = L/2$. So, after the complete relaxation, the height will again reduce to $L/2$. The reduction of the height Δh vs t is displayed for different, L=1000, 2000, and 3000 in Fig. 5(a). Initially, the relaxation seems to be slower than what we have seen with random distribution and asymptotically it seems to have a slight curvature. In order to extract exponents, we did simulations for t up to 15 000. As shown in Fig. 5(b), the slope is impressively linear in t and z=1. The total relaxation time $T(L)$ also remains linear with z'=1 [Fig. 5(c)]. Since less voids are present near the top with this distribution, the initial slowing down is understandable. We have also examined the effect of fluctuations in the void distribution by adding the random number to the existing distributions, namely, $\rho(z,0) = 1 - z/L + \zeta$ with $0 < \zeta =$ random number < 1. We have found no change in the exponents z and z'.

(c) Boltzmann distribution. Distribution is given by $\rho(z,0) = \exp(-\alpha z)$ with $\alpha = 0.01$. In this case, we have also found linear behavior and thus the results will not be presented here. It is quite interesting that within the

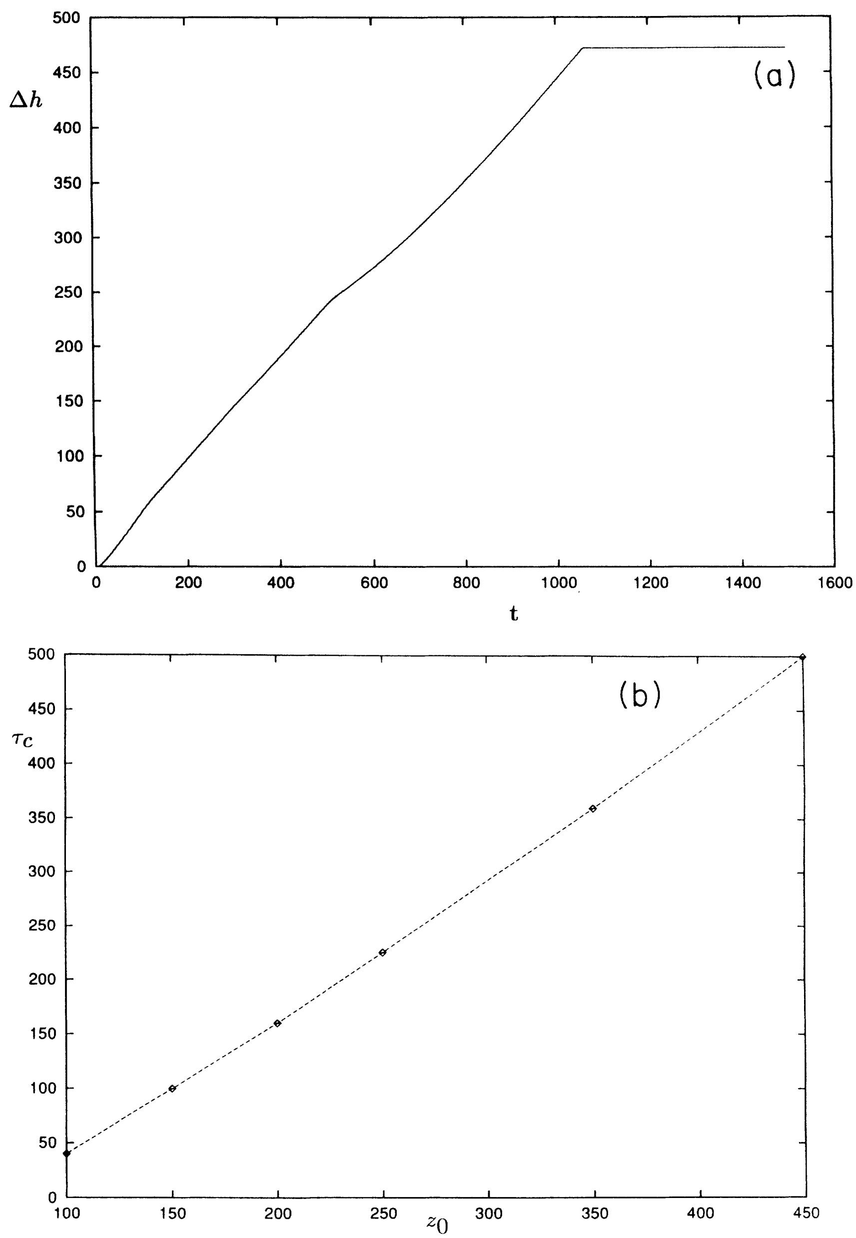

FIG. 4. (a) Relaxation of the height with random distribution in the presence of a barrier at $z = z_0$, with the fixed density $\rho(z_0,t) = 0.05 << 1$. The relaxation slows down beyond the crossover time, τ_c, but overall the process is still linear. (b) The crossover time, τ_c as a function of the position, z_0. The behavior is linear.

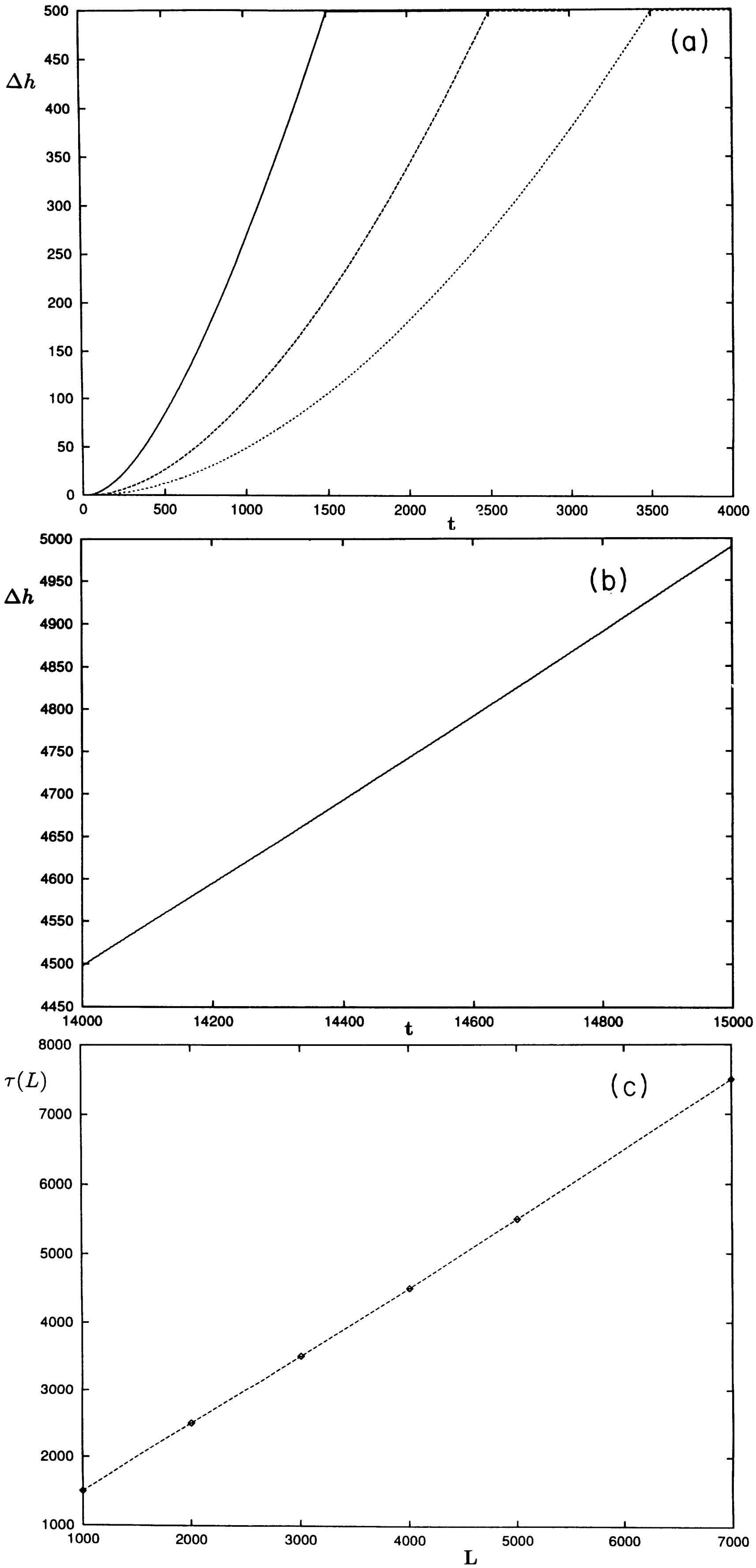

FIG. 5. (a) Reduced height $\Delta h(t)$ as a function of t with linear distribution for three different L=1000, 2000, 3000 far from left. Initially, the relaxation is slow because less voids are present near the top. (b) Long time behavior of the height reduction against time t. The slope is one giving $z = 1$. (c) The total relaxation time, $T(L)$, is plotted as a function of L. The slope is one and hence z'=1.

context of the diffusing void model, the relaxation of the grains in a simple one-dimensional pile is independent of the initial void distributions and the reduction of height is *linear* with the dynamic exponent $z = z'=1$, even though the governing equation is nonlinear. In order to trace the origin of this linear behavior, we have examined the time dependence of the rising of a single void from the bottom. The ascending time was measured as a function of the height of the pile L. Not surprisingly, it is again linear.

We now search for the origin of the linear behavior by examining the steady state solution of Eq. (6). Since the asymptotic behavior is determined mainly by the first term of (6), we expect that the following equation with the modified diffusion term should yield the same asymptotic result:

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial z}[V_0\, \rho\,(1-\rho)] + D\,\frac{\partial}{\partial z}\left(\frac{\partial \rho}{\partial z}\right), \tag{8}$$

The above equation assumes the traveling wave solution of the form: $\rho(x,t) = \rho(x-vt)$. In the moving frame, ρ satisfies

$$D\, d\rho(x)/dx = -V_0\rho(x)[1-\rho(x)] + v\rho + c, \tag{9}$$

where c is a constant that must be determined by the boundary conditions. Equation (9) can be solved exactly with two boundary conditions, $\rho(x=-\infty) = \rho_{-\infty}$ and $\rho(x=+\infty) = \rho_\infty$. Going back to the laboratory frame, we find

$$\rho(x,t) = -(v-V_0)/2V_0 + Q\, \tanh[(V_0/D)\, Q(x-vt)], \tag{10}$$

where

$$Q = (\rho_\infty + \rho_{-\infty})/2; \tag{11}$$

$$v = V_0[1-(\rho_{-\infty}+\rho_\infty)]. \tag{12}$$

One way of understanding the existence of this traveling solution is to use the language of dynamical systems theory. Equation (9) has two fixed point, $\rho_{-\infty}$ and ρ_∞. The former one is unstable and the latter is stable. Thus, the trajectory will move from $\rho_{-\infty}$ to ρ_∞ as x moves from $-\infty$ to ∞, which is the solution given by Eq. (10) (Fig. 6). In the laboratory frame, the traveling wave solution, Eq. (10), moves with the speed v given by Eq. (12). One might impose the solvability condition at the stable fixed point to determine the speed v as a function of external parameters as was commonly done in dendritic solidification and viscous fingers. It is tempting to suggest that if we include the threshold condition to the dynamic equation (6), then there might exist no v that satisfies the solvability condition. If so, then the traveling wave solution does not exist, leading to the different universality class. Note that the interface thickness is determined by the ratio V_0/D. Also note that the speed of the traveling wave is independent of the diffusion constant, which justifies the use of Eq. (8) rather than (6), and is determined by the sum of the density at the end points, $\rho_{-\infty}$ and ρ_∞. If the sum is one as in the case considered here, then v=0 and the interface does not move. This is indeed true if the system is infinite, but if the system is finite with length L, the situation is different. To see this more clearly, let us consider the time dependence of the total density in the system,

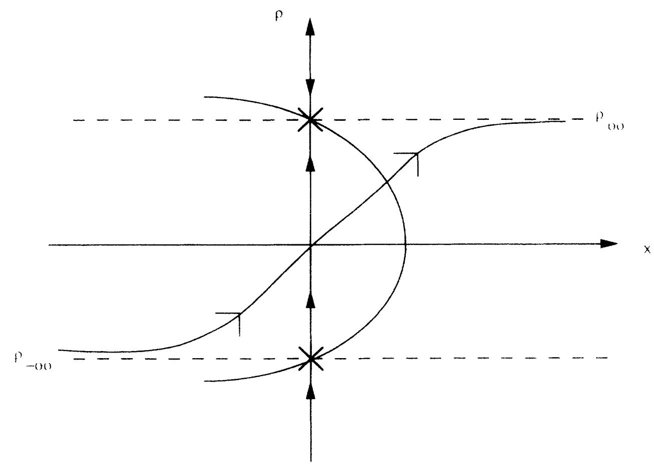

FIG. 6. Equation (9) has two fixed points: the stable one at $\rho_+ = \rho_\infty$ and the unstable one at $\rho_- = \rho_{-\infty}$. The parabola in the Fig. is the right hand side of Eq. (9) and the upper and lower lines refer to the boundary conditions of the pile at $x=\infty$ with ρ_∞ and $x=-\infty$ with $\rho_{-\infty}$. The solution of (9) is the trajectory that moves from ρ_- to ρ_+ with the constant speed V_0 as x changes from $-\infty$ to ∞.

$$\rho_T = \int_{-\infty}^{\infty} \rho(x)dx.$$

Direct integral of the Eq. (8) yields

$$d\rho_T/dt = [-V_0\rho(1-\rho) + D\partial\rho/\partial x]|_{-\infty}^{\infty}. \tag{13}$$

If the system is infinite, then both terms in the right hand side of (13) vanish and the total density is conserved and the interface does not advance. However, for the finite system, the first term vanishes but the gradient term does not. Hence, the total void inside the system changes. The difference in the density moves out into the sea of voids and the reduction of the height occurs. What we have shown here is that the origin for the linear behavior in the dynamic relaxation of a granular pile under tapping, within the context of the diffusing void model, lies in the existence of the steady state traveling wave solution, which advances with the constant speed.

There is one simulation where such a linear behavior has been observed in the context of the segregation of grains. The authors of Ref. [12] carried out the molecular dynamics simulations for the segregation of larger particles among smaller ones under shaking and they have reported that the upward displacement of the larger one is linear in time. It remains to be seen whether real experiments would reveal such a linear behavior.

IV. STICK-SLIP RELAXATION AND ITS RELATION TO THE TRAFFIC PROBLEM

In this section, we make an attempt to modify the diffusing void model considered in the previous sections by taking into account fluctuations and long range correlations among voids. In the previous sections, we have assumed that the void velocity V_z depends only on the density right above it and hence obtained Eq. (6), which consists of the two terms: $V_z = V(\rho) + V_d$. Here $V(\rho) \equiv V_0(1-\rho)$ might be viewed as the density-dependent average void velocity; the form $(1-\rho)$ is not inconsistent with the observation that voids repel each other and hence the average velocity of the void reduces in the high density regime. The second term $V_d \equiv D\, d\rho/dz$ is a simple diffusion term, which needs some critical examination. Note that Eq. (6) does not take into account fluctuations in the void velocity resulting from, e.g., long range correlations. For example, suppose the grains are very rough. Then the void size distribution produced by such rough grains presumably remains fairly uniform and each void is expected to be a robust object. In that case, such a robust void might change its shape during the time the motion is taking place, but it does not entirely lose its identity as a round object and can subject itself to compression rather than losing its identity via the continuous diffusion. In this limit, the void first adjusts its velocity to the mean value $V_0(1-\rho)$, dictated by the density of the immediate neighborhood surrounding it. At the same time, as a compressible object, it does also have to adjust its speed to the density gradient ahead of it. Hence, in such cases, it would be presumably more appropriate to write down the *time dependent* equation by which voids can adjust their speeds. In addition, the void is not a rigid body but a hydrodynamic variable, so we should replace the time derivative by the total (hydrodynamic) derivative. We thus take into account all these factors, and obtain the modified equation for the void velocity:

$$\frac{\partial V_z}{\partial t} + V_z \frac{\partial V_z}{\partial z} = \gamma\,[V(\rho) - V_z] - \frac{c_0^2}{\rho}\frac{\partial \rho}{\partial z}. \qquad (14)$$

Note that the left hand side is simply the hydrodynamic derivative term while the first term on the right hand side represents the fluctuations of the void velocity around its mean value with γ^{-1} the adjusting time scale. If we set the left hand side of (14) to zero, it effectively reduces to (5) except the additional factor, ρ^{-1}. The mean velocity profile $V(\rho)$ is a smooth and decreasing function of ρ, although its exact form is not known. In this paper, we assume the following simple form:

$$V(\rho) = \frac{\alpha}{1+\beta\rho^2}, \qquad (15)$$

with α and β being constants. Note that in the diffusing void model, this term is proportional to $1-\rho$, which has a sharp cutoff at $\rho = 1$. Due to the numerical instability with such a sharp cutoff, we have instead chosen the form (15), which seems to fit the experimentally determined $V(\rho)$ for the real traffic flow [32]. But certainly one might try a different form as long as it decays as a function of ρ.

It is of interest that with the identification of ρ and V_z as the density and the speed of cars in a highway, respectively, Eq. (14) along with the continuity equation (3) are precisely those used in the traffic problem [32]. In this case, there exist data for the mean velocity profile of the cars as a function of the density [32], to which Eq. (15) seems to be a reasonable approximation. In Eq. (14) c_0^2 is the dispersion in the speed distribution, which was measured to be of the order of 25 miles per hour in the highway considered in Ref. [32]. In the context of the traffic problem, the justification of using the time dependent equation (14) rather than simple static one such as (5) is not difficult to make because the driver will do his best to avoid collisions in the highway, first by simply adjusting his speed around the mean value $V(\rho)$ and second by either slowing down or speeding up depending on the density gradient of cars ahead of him and/or to obey the traffic signs on the highway. We assume that the moving void in the granular assembly made of rough grains is fairly smart much the same as the driver in the traffic problem: each void does not simply diffuse out, but is now subject to *compression.* The number of cars passing by the toll gate in the highway then corresponds to the the density $\rho(L,t)$ at the top layer of the granular pile.

Kühne [32] studied the traffic equations and identified the stability parameter, $\lambda = -1 - (\rho_0/c_0)(\partial V/\partial \rho)|_{\rho_0}$. He showed that for $\lambda > 0$, the homogeneous solutions become unstable and subsequently argued that they may bifurcate to either traveling solutions or periodic stop-start solutions. Recently, Kerner and Kohnhauser have added a viscous term to the traffic equation and their numerical solutions indeed have revealed the existence of the traveling shock wave [33]. Neverthless, to the best of our knowledge, Eq. (14) has not been studied in detail, and we have made our own numerical investigations of Eqs. (14) along with (15) in order to gain insight into the relaxation of granular assembly.

Before presenting numerical solutions, we first describe a subtlety in dealing with the compression term with open boundary conditions. Consider a one-dimensional column of height $h_0 = L$. We now discretize it into L positions, each of which is labeled by the index i $(= 1, ..., L)$. If we use the same boundary conditions as was done to solve Eq. (7), namely, setting $\rho(0) = 0$ and $\rho(L+1) = 1$ with random initial distribution, then the density at each position quickly approaches zero and we encounter a numerical problem due to the factor ρ^{-1} in the compression term. We avoid this numerical instability by employing the following boundary conditions at the two end points, $z = 1$ and L. First, we put a source beneath the bottom at $z = 0$ and a sink right above the top at $z = L+1$ with $L = 100$. The initial values of $\rho(i,0)$ are chosen randomly such that the initial average void density is $\langle\rho\rangle = 0.5$. We also take $V_z(i,0) = 0.5$, together with the fixed source density $\rho(0,t) = \epsilon$ $(0<\epsilon<1)$ and $\rho(L+1,t) = 1$. This corresponds to the situation that constant flux of grains is fed into the tube while the grains escape through the bottom. Hence, in this case, the density $\rho(L,t)$ describes the number of grains flowing out of the tube at given time t, or equivalently, it is the density of voids that arrive at the top. Note that this is *not* the accumulated density at

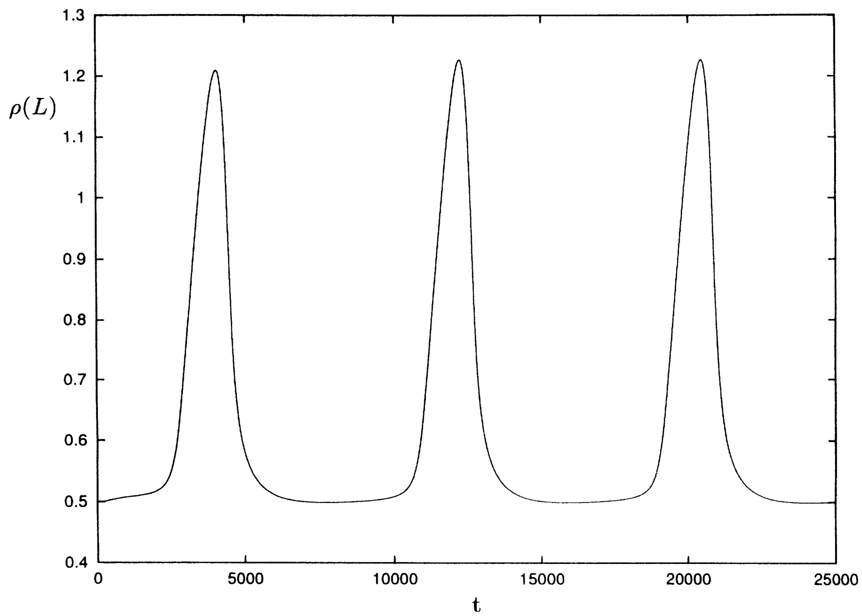

FIG. 7. The density at the top obtained from the traffic equations (3) and (14) with a source and a sink. Note the appearance of the periodic solution.

the top.

Further, we have employed the following discretization at the end points:

$$\left.\frac{\partial\rho}{\partial z}\right|_{z=L} = \frac{\rho(L)-\rho(L-1)}{\Delta z},$$

$$\left.\frac{\partial\rho}{\partial z}\right|_{z=1} = \frac{\rho(2)-\rho(1)}{2\Delta z},$$

while for points inside ($i = 2, ..., L-1$), we have used the standard discretization,

$$\left.\frac{\partial\rho}{\partial z}\right|_{z=i} = \frac{\rho(i+1)-\rho(i-1)}{2\Delta z}.$$

Equipped with these, we have solved the traffic equations numerically for $\lambda > 0$ and determined the density $\rho(L,t)$ at the top layer. The time step and the step size have been chosen as $\Delta t = 0.1$ and $\Delta z = 1.0$, respectively.

Figure 7 displays the numerically obtained density at the top as a function of time, for $\alpha = 1.2$, $\beta = 4.2$, and the source density $\epsilon = 0.6$. Note the appearance of the periodic solution. Even though a steady and uniform number of voids are supplied by the source at the bottom, these voids do not arrive at the top uniformly; rather, clusters of voids arrive periodically. In order to see what is actually happening inside the tube, we have examined the density profile as a function of the position at various times, which is shown in Fig. 8. Initially, the density distribution is random with the mean value $\langle\rho\rangle = 0.5$. After certain time steps, this random distri-

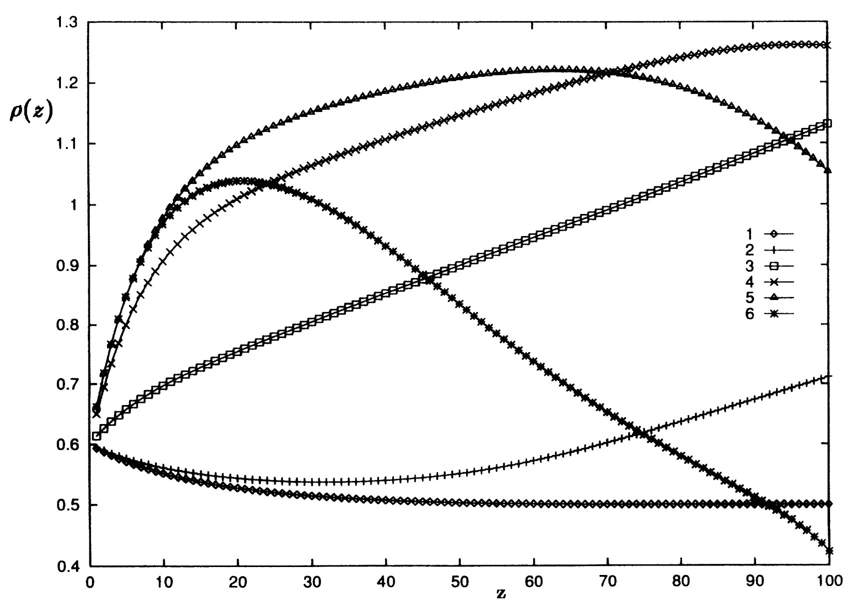

FIG. 8. The snap shots of the density profile inside the tube at various times. The numbers refer to the time sequence.

bution quickly settles to a uniform one inside the tube, which is followed by the density profile developing a concave yet positive gradient toward the sink and the continuous growing of the density at the top. It is during this period that the voids are being compressed and accumulated near the top. While the drift term in Eq. (14) is pouring out constant flux of voids into the sea (note that the background density in Fig. 7 is finite, around the mean value 0.5 of the initial void distribution), it appears that a certain fraction of voids, presumably those that are entering at the bottom, cannot simply sneak out into the sea, which is mainly due to the absence of the diffusion term in Eq. (14). This is similar to the situation of a void moving up in water, due to the buoyant force. If the void is small and hence the buoyant force is not large enough to break the energy barrier at the air-water interface, it is stuck right below the surface: In order for the void to come out of the water, it must create an extra surface, which requires extra surface energy. In the case considered here, while the physical origin of such accumulation of voids at the free surface is not obvious, we offer the following explanation. As we have mentioned in the beginning of this section, we are dealing with diffusing voids that are robust, compressible, and able to preserve their overall topology as round objects. In addition, we recognize that the roughness at the grain boundary induces effective attractive interactions among grains, which are the source of the energy barrier at the grain-air boundary. When enough voids are accumulated at the top, they form a shock or a pulse which can collectively break the energy barrier, and push themselves out into the sea of voids. So the void density at the sink will suddenly increase when such a shock is released at the top. At this point $\rho(L,t)$ starts to decrease and the density profile inside the tube also reverses its shape from concave to convex, shifts its maximum toward the sink, and then slowly disappears as one cycle is completed. (See Fig. 8.)

The appearance of the periodic pulse at the top is a clear indication that clusters of voids in the form of shocks or pulses arrive at the top periodically. Hence, in the regime where the traffic equation makes more sense than the diffusing void equation, we anticipate that the reduction of the height of granular pile will be sudden and discontinuous, which we term the stick-slip relaxation (Fig. 9). Experimentally relevant parameters might be the period and the width of the pulse as a function of the control parameter, λ. There does not exist a general principle to determine the period for arbitrary λ. Near the onset of the instability, however, one might argue that the period is determined by the imaginary part of the growth rate ω. In the limit of small λ, the linear stability analysis for the unstable branch yields the following dispersion relation:

$$\omega = 2k^2\lambda/(1+4k^2) + ik[(1-v_0) + \lambda/(1+4k^2)], \quad (16)$$

from which we find that the period, T, of the pulse is

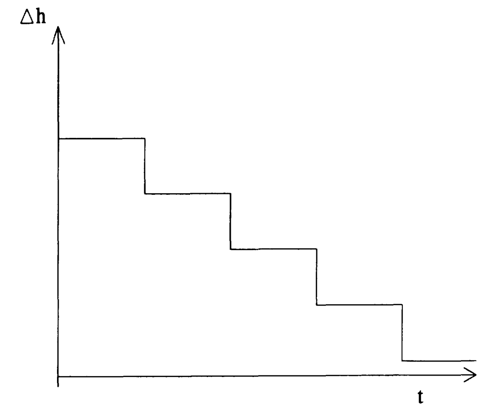

FIG. 9. Schematic picture of the stick-slip relaxation. When enough voids are accumulated near the top, they form a shock or a pulse which collectively breaks the energy barrier and results in a sudden reduction in the height.

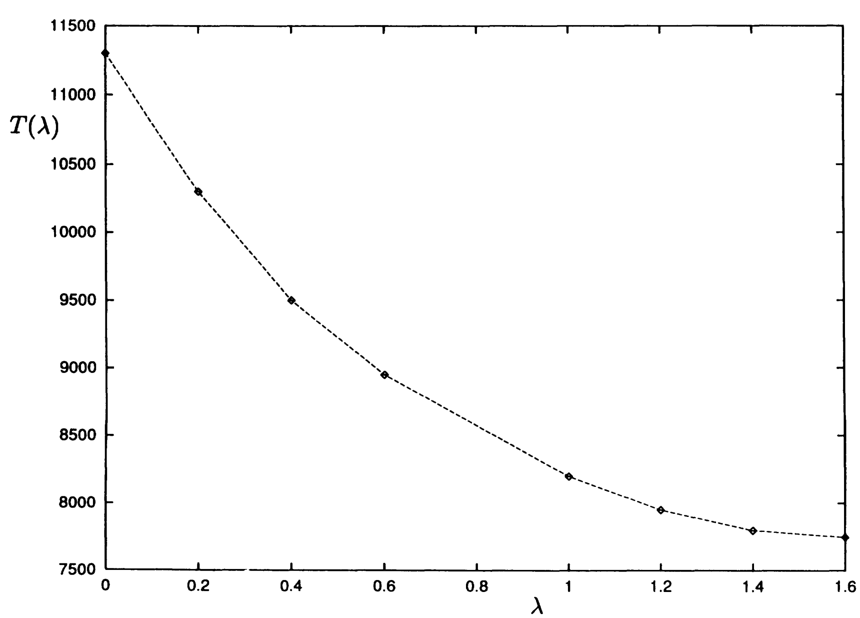

FIG. 10. The dependence of the period of the pulse on the control parameter, λ. It decreases as λ increases, consistent with the prediction of the linear stability theory [Eq. (17)].

given by, $T = 2\pi/\Omega$, where

$$\Omega = k[(1 - v_0) + \lambda/(1 + 4k^2)], \qquad (17)$$

The above equation predicts that the period of the pulse with a well defined wave number k is inversely proportional to λ, which holds only near the onset. Far beyond the onset, a single pulse will contain many Fourier modes, in which case due to the dispersion the period will not follow the simple formula given by Eq. (17). Numerically, we find that the periodic pulse solution exists at and beyond $\lambda_c = 0$ for which case the prediction of the linear stability does not necessarily hold. We, however, have found that the linear dependence of the period on the control parameter λ seems to persist as shown in Fig. 10. The complete knowledge regarding the existence of a pulse solution such as a soliton and the dependence of its speed, period, and width on control parameter λ, requires a detailed nonlinear analysis of the traffic equations and we will report it in the future work.

V. SUMMARY

In this paper, we have examined the dynamic relaxation of a granular pile in a confined geometry under repeated tapping. Within the context of the diffusing void model, the relaxation process is algebraic with the dynamic exponent $z = z' = 1$, which is robust against perturbations and is independent of the initial void distributions. We have shown that the existence of the steady state traveling solution is responsible for such a linear behavior. The experimental observation of the slow relaxation [24] of the grains under tapping requires more studies. We have also shown that a simple modification of the diffusing void model leads to the traffic equations, which predicts the discontinuous relaxation of the grains. We were informed by Jaeger [24] that this type of discontinuous relaxation indeed occurs when the grains are *rough*; the underlying physical origin of such relaxation might be understood as discussed in Sec. IV. For moderate amplitudes of the tapping, molecular dynamics data also reveal such discontinuous relaxation [34]. In this regime, the grains move little by little until a sudden hexagonal packing is formed locally, which generates a massive number of voids. This process repeats until all the grains relax to the ground state of the complete hexagonal packing. Finally, we emphasize that our approach here has focused only the geometrical aspect of the packing of grains. The future work must take into account, in particular, the threshold condition for initiating the dynamical process, avalanches, dynamics of different void size along with the interactions, elaborate friction laws between grains, or the metastable nature of the ground state of the grain configuration.

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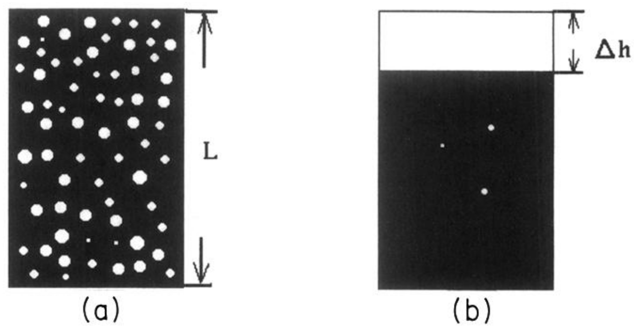

FIG. 1. Schematic picture of the relaxation of a granular pile under repeated tapping. (a) Grains are confined in a quasi-one-dimensional tube whose initial height is L. Since the grains (black) are randomly packed, there exist numerous minute voids inside (white circles). The top layer of the pile meets the sea of voids. (b) Under repeated tapping, grains relax and the voids move up, which are then accumulated at the top. The reduction of the height Δh is proportional to the total accumulated void density at the top layer [See Eq. (1)]. Note that the massive black also contains regular as well as irregular voids even in the totally relaxed configuration, which must be subtracted in the stricted sense.